AAS 93-635

JACOBI'S INTEGRAL AND ΔV-EARTH-GRAVITY-ASSIST (ΔV-EGA) TRAJECTORIES

Theodore H. Sweetser, Ph. D.†

The ΔV-Earth-Gravity-Assist trajectory technique offers powerful advantages in maximizing payload mass, but the conventional "explanation" of how it works leaves the following paradox unexplained: by slowing down at the slowest point in the trajectory, far more energy is gained than an equal speed-up at the fastest point. A better explanation results from consideration of Jacobi's integral, which is calculated in the rotating frame in which the Sun-Earth line is fixed. In this frame the deep-space maneuver magically transforms into one which increases the velocity at the point where the velocity is already maximum.

This perspective on the ΔV-EGA has two advantages. Firstly, it allows a simple calculation of the ratio between the magnitude of the deep-space maneuver and the resulting increase in the effective departure velocity. Secondly, it has suggested possible variations in this technique. In particular, a new kind of lunar transfer trajectory from low Earth orbit is presented which has lower ΔV requirements than the Hohmann transfer.

INTRODUCTION

One of the most powerful tools in the trajectory designer's bag of tricks is the ΔV-Earth-Gravity-Assist (ΔV-EGA). This tool, first introduced by G. R. Hollenbeck[1] at this conference in 1975, uses a maneuver in deep space in such a way that it has a greatly magnified effect on the final departure velocity from Earth. Typically, a deep space maneuver of 0,5 km/s can increase the departure velocity by 2 km/s or more, where the only cost incurred is an increased flight time.

On the ΔV-EGA trajectory (see Figure 1) a spacecraft leaves Earth on, for example, a two-year heliocentric orbit. At the aphelion of that orbit a

† Mathematician, Member AAS, AIAA, Member of Technical Staff, Jet Propulsion Laboratory, 4800 Oak Grove Dr., Pasadena, CA, 91109. Phone: (818) 354-4986,

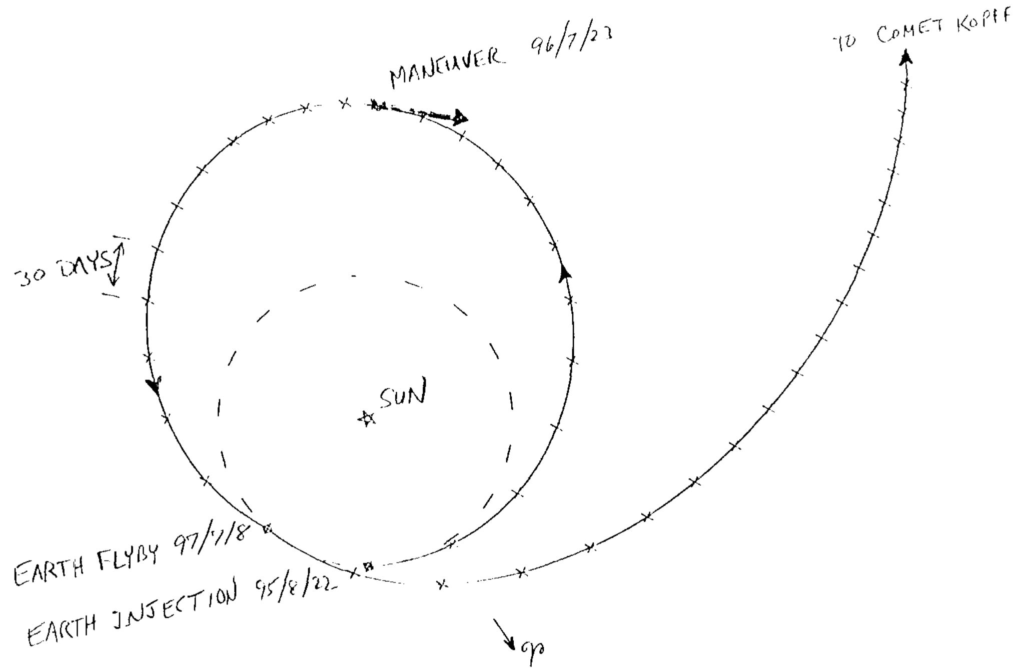

Figure 1 **Inertial ecliptic plane view of an exterior ΛV-Earth-Gravity-Assist trajectory to Comet Kopff.**

maneuver is performed which reshapes the orbit to lower its perihelion. The orbit timing is arranged so that when the spacecraft then crosses Earth's orbit (either before or after periheli on) it encounters Earth in a gravity assist maneuver. The advantage of the ΛV-EGA is that the increase in perigee velocity from the launch to the encoun ter is much greater than the velocity change at the deep-space maneuver. The resul ting heliocentric energy is greater than in the initial two-year orbit ant] again the increase is more than could be obtained from the deep-space maneuver alone.

Two aspects of the ΛV-EGA are counterintuitive (as is often the case in orbi tal mechani es). One is that the deep-space maneuver which sets up the heli ocentric energy gai n actual] y reduces the heliocentric energy - the spacecraft s] ows clown at aphel i on to move the perihelion closer to the sun. The second aspect is perhaps more puzzling to the experienced trajectory designer (who is used to the occasional] y paradoxical behavior of orbits). 1 n general, the most efficient time to change orbital energy is when the veloci ty is highest, i.e., at periapse. 1 n the ΛV-EGA, however, the clmp-space maneuver is most effective when done where the velocity is lowest, i.e., at aphelion.

The conventional explanation of the ΔV-EGA ignores these aspects as follows: it is easiest to reshape an orbit where the velocity is lowest; the more the orbit shape is changed, the greater the angle between the spacecraft orbit and Earth's orbit where they cross; the greater the angle, the greater the difference between the spacecraft's velocity and Earth's velocity at encounter; this velocity difference when aligned with Earth's velocity by the gravity assist gives us our final heliocentric energy. All this is true and is fine as far as it goes. Nevertheless, there remains an element of mystery in the ΔV-EGA because of the paradoxes discussed above and because there is no direct way in the conventional explanation to relate the magnitude of the deep-space maneuver to the final gain in heliocentric energy. This mystery is deepened by the fact that an analogous trajectory in the Earth-Moon system (Figure 1 with the Earth replacing the Sun and the Moon replacing the Earth). This trajectory leaves the Moon on a two-month orbit with a maneuver at the apogee but does not show *any* advantage with respect to the final Earth-relative energy over increasing the lunar departure velocity by an equal amount. '1'here is nothing in the conventional explanation which would predict that a tool which works in the Sun-Earth system fails in the Earth-Moon system.

JACOBI'S INTEGRAL EXPLAINS ALL

The reason for the apparent mysteriousness of the ΔV-EGA trajectory is that th c discussion above considers the trajectory as a series of two-hod y problems: l'larth/spacecraft for launch, Sun/spacecraft for initial orbit and deep-space maneuver,]CarLh/spacecraft for gravity assist maneuver, and Sun/spacecraft for final orbits. 1 3ut the ΔV-EGA is very much a creature of the three-body problem, in which it is not appropriate to base an analysis on energy. Instead, we must turn to the three-body analog of energy, Jacobi's integral.

If the Earth traveled in a circular orbit around the Sun and the only accelerati ons experienced by a (mass] css) spacecraft were caused by the central gravity of the Earth and Sun, then J acobi's integral[2,3]

$$C = -v^2 + \omega^2 \rho^2 + \frac{2\mu_e}{r_e} + \frac{2\mu_s}{r_s} \qquad (1)$$

is a constant al ong the spacecraft's trajectory, where v is the magnitude of the **rotating-coordinate velocity**, which is the velocity of the spacecraft in a

Table 1.

CONSTANTS

a	$= 149597870.$ km	mean semi-major axis of the Earth's orbit = 1 AU
μ_S	$= 1.327124 \times 10^{11}$ km^3/s^2	Gravitational constant times the mass of the Sun
μ_e	$= 398600.5$ km^3/s^2	Gravitational constant times the mass of the Earth
ω	$= 1.990987 \times 10^{-7}$ rad/s	mean angular rotation rate of the Earth-Sun system

threw dimensional coordinate system centered at the Earth-Sun barycenter and rotating with the Earth-Sun system, p is the distance from the barycenter to the projection of the spacecraft's position onto the Earth-Sun orbit plane, r_e is the distance from the spacecraft to the Earth, and r_S is the distance from the spacecraft to the Sun. (Definitions for ω, μ_e, and μ_S arc given in Table 1.)

In a two-body problem, energy is a constant function of position and the magnitude of the inertial velocity. In the circular restricted three-body problem, Jacobi's integral is a constant function of position and the magnitude of the rotation-relative velocity. For our purpose% here we may consider a maneuver to be an instantaneous velocity change which dots not affect position. Thus, while an energy change is maximized for a maneuver if the maneuver is done when the inertial velocity is greatest (at the periapse of a conic), a change in Jacobi's constant is maximized if a maneuver is done when the rotating-coordinate velocity is greatest.

This is the key to understanding the ΔV-EGA. The trajectory of Figure 1 is replotted in Figure 2, but this time in the rotating frame which keeps the Sun-Earth line fixed. The figure shows graphically how the deep-space maneuver is in fact done when the magnitude of the rotating-coordinate velocity is greatest and is done in the direction of the rotating-coordinate velocity, Furthermore, the consequent change in Jacobi's constant can be used to estimate the velocity increase from launch perigee to encounter perigee which results from the deep-space maneuver, so the magnifying effect of the ΔV-EGA can be calculated.

A NUMERICAL EXAMPLE

Of course the real world is not a circular restricted three-body problem. Nor has a straightforward ΔV-EGA trajectory been flown in a space mission. But ΔV-EGA trajectories have been carried as baseline trajectories during the design process of several missions. In particular at one time the baseline

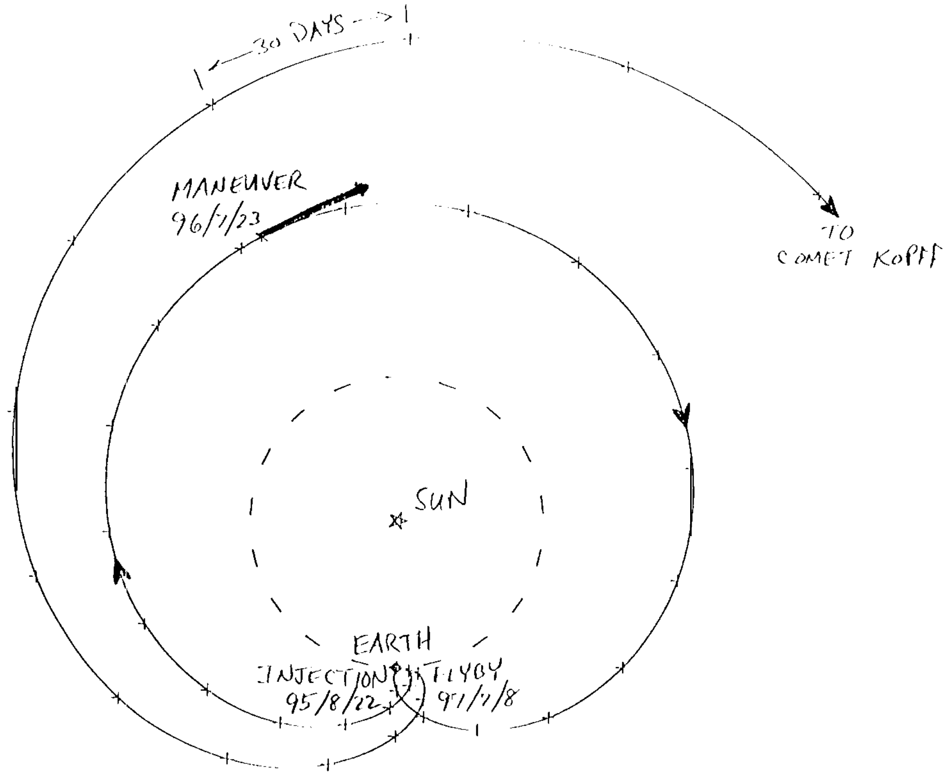

Figure 2 Rotating ecliptic plane view (fixed Sun-Earth line) of the exterior ΔV-Earth-Gravity-Assist trajectory shown in Figure 1,

trajectory for the Comet Rendezvous/Asteroid Flyby mission (CRAF)[4] used a two-year ΔV-EGA which had a clccp-space maneuver of 0.6 km/s and a flyby perigee velocity which was 2.2 km/s greater than the insertion perigee velocity (sew Figure 1). 1 ret's compare this to an estimate obtained by using J acobi's constant.

We start by assuming the Earth travels in a circular orbit around the Sun according to the constants in Table 1. A two-.ycar orbit which is tangent at perihelion to Earth's orbit has an aphelion distance of 2.1'75 AU. At that distance, a point fixed in the rotating Earth-Sun system has an inertial velocity of 64.78 km/s (= 2.175 AU) in the direction of the rotation; equivalently a point fixed in inertial space at that distance has a rotating-coordinate velocity of equal magnitude but in the opposite direction. Since the spacecraft speed at apheli on is 16.03 km/s, the spacecraft's rotating-coordinate velocity there is 48.75 km/s.

A point fixed close to Earth, say at 170 km altitude, has a negligible rotating- coordinate velocity in the Earth-Sun rotating system. Thus for the launch and encounter, the spacecraft's rotating-coordinate velocity is essentially the same as its Earth-relative velocity regardless of the orientation of the hyperbola, This is 12.15 km/s at 170 km altitude on a hyperbol a launching into a two-year heliocentric orbit.

To calculate the change in Jacobi's integral resulting from a velocity change, we substitute w Δv for v in equation (1) and compare the result to get

$$-\Delta C = 2v\Delta v + (\Delta v)^2 \qquad (2)$$

so that for small **Av** we see that ΔC ***is*** roughly proportional to the rotating-coordinate velocity. For the case analyzed here this gives about a magnification factor of 4 (≈4 8,75/12,15), in good agreement with the data. More precise] y, if v_a is the rotating-coordinate velocity at aphelion and v_E is the rotating-coordinate velocity at perigee, we have

$$2\, v_E \Delta v_E + (\Delta v_E)^2 = 2\, v_a \Delta v_a + (\Delta v_a)^2 \qquad (3)$$

or

$$(\Delta v_E)^2 + 2 \cdot 12.15 \Delta v_E - 2 \cdot 48.75 \cdot 0.6 - 0.6^2 = 0 \qquad (4)$$

so that $\Delta v_E = 2.22$ km/s, in even better agreement with the data.

Now we can see why an analogous trajectory departing outward from the Moon fails to have any advantage. The rotating-coordinate velocity at the apogee of a two month orbit (whose perigee is tangent to the Moon's orbit) is less than the perilune velocity at departure, which is 2.3 km/s. The Moon's gravitational attraction is too large compared to the Earth's for a twice-period maneuver orbit to set up an advantageous maneuver. A threw-month orbit woul cl offer a slight advantage, though.

VARIATIONS ON AV-GRAVITY-ASSIST TRAJECTORIES

The example above was a two-year ΔV-EGA, but of course there is nothing to constrain the initial orbit to have a two-year period. A thrm-year orbit wound do as well and in fact, as this analysis implies, gives a greater magnifi cation of the deep-space maneuver. A less commonly considered alternative is a 1 .5-year orbit whi ch encounters Earth after three years.

Another variation is simply to reverse the trajectory, making it an arrival rather than a departure. 1 n this reverse ΔV-EGA trajectory a spacecraft, woul d come in tangent to Earth's orbit where a gravi t-y assist would 1 ower both aphelion ant] perihelion. Then a maneuver at aphelion would raise perihelion back up to tangency with Earth's orbit for an arrival with a lower V_∞ than at the first encounter. The reverse ΔV-EGA would be effective at Earth for sample return missions from outer planets, comets, or asteroids. But it's usefulness is not confined to arrival at Earth. As C.-W. Yen[5] showed at this conference in 1985, such a trajectory technique is very useful for reducing the rendezvous requirements of M ercury orbiter missions. From the point of view of an anal ysis based on Jacobi's integral, the maneuver is s] owing down the rotating-coordinate velocity at the point where that velocity is maximum. Calculation of the ratio between the maneuver and the savings in rendezvous AV would be the same as in the previous section.

Arc there other variations? The above analysis has shown us that all that is really necessary for a ΔV-Gravity-Assist (ΔV-GA) trajectory is that a spacecraft have successive encounters with a body where the orbit between encounters contains a point where its rotating-coordinate velocity is greater than at encounter periapse. in the cases considered so far, the intermediate orbit is outside the body's orbit so that in the rotating frame it looks retrograde with a maximum velocity at apoapse.

This leads us to suspect the existence of a new type of AV-GA trajectory. The ΔV-EGAs above all start with orbits larger than Earth's orbit; let's call them exterior ΔV-EGA trajectories. What about interior ΔV-EGA trajectori es, that start off with orbits smaller than Earth's? For example, if a spacecraft starts off in a 2/3- year orbit it leaves with very nearly the same velocity relative to Earth as in a two-year orbit but in the opposite direction. At perihelion the spacecraft has a rotating-coordinate velocity of 31.32 km/s so a magnification factor of about 2 1/2 is possible for this ΔV-EGA. Figure 3 shows such an interior ΔV-EGA used for a transfer from Earth to Mercury. in this case, a AV of 1.4 km/s at perihelion results in a velocity increase of 3.2 km/s from injection perigee to flyby perigee. For this type of trajectory one of the paradoxes discussed earlier has disappeared since the maneuver is done at perihelion. The other paradox remains - after speeding up at perihelion the ultimate solar orbit is smaller.

Such interior ΔV-EGA trajectories were mentioned by Hollenbeck[1] when he first introduced the ΔV-EGA technique. He considered them (in the context of outer-planet missions) only briefly, because of the fortuitous circumstance that if the period of the intermediate orbit is less than 4/5 year then a gravity assist at Venus becomes possible, giving a free ΔV. Indeed, such a trajectory is being used by Galileo, where the "second" Earth encounter is actually done twice over because the V_∞ at Earth starts out

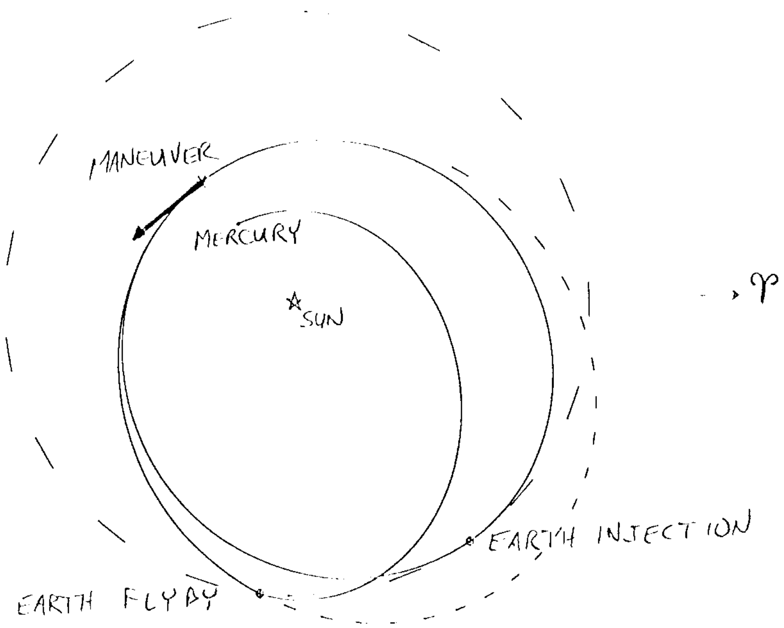

Figure 3 **Inertial ecliptic plane view of an Interior ΔV-Earth-Gravity Assist trajectory to Mercury.**

opposite to Earth's velocity direction and has to be turned around to align with it in order to get a departure to an outer planet (Jupiter in this case).

Such extreme bending of the V_∞ vector is not needed if the interior ΔV-EGA is being used to get to an inner planet such as Mercury. Once again, however, the availability of Venus obviates the need for a propulsive maneuver. Furthermore, a return to Earth is not desired since a rendezvous at Mercury has a lower velocity after arrival from Venus than after arrival from Earth. What remains is precisely the kind of trajectory discussed by C.-W. Yen[5]. Thus it seems that interior ΔV-GA trajectories have no useful application to inner planet missions. In reverse they could be used for reducing rendezvous ΔV in Mars or outer-planet missions, but then the flight times become prohibitive (a 1:2 resonant rendezvous with Jupiter, for example, would add 11 years to the flight time, and that's the best possible). But there is one application where conditions are favorable for a ΔV-GA trajectory,

'1 HE REVERSE INTERIOR ΔV-LUNAR-GRAVITY-ASSIST

The lunar transfer problem has just the right situation for this application. The spacecraft starts out in an orbit relatively close to the primary and injects into a Hohmann transfer to the Moon. instead of rendezvousing immediately, however, the spacecraft flies by so that the gravity assist at the Moon raises both perigee and apogee. Then a maneuver at perigee lowers the apogee back to tangency with the Moon's orbit, with a net savings depending on the rotating-coord inate velocity at perigee and the amount of apogee change needed.

As it turns out, the relatively large gravitational attraction of the Moon results in high perilune velocities at rendezvous. The rotatinS-coordinate velocity at perigee is higher only when the Perigee is close to Earth. Furthermore, the Moon's orbit has significant eccentricity, which complicates the analysis. Using conies the result is that for rendezvous at the Moon's perigee, only intermediate orbits between 1/3 and 2/5 of the M eon's period can be used; at the Moon's apogee the period can be increased to 1/2. We are now faced with two counteracting factors. On one hand, the lower the perigee the higher the rotating-coordinate velocity so the higher the magnification multipli er that can be applied to the maneuver. On the other hand, a lower perigee raise caused by the lunar flyby corresponds to a lower apogee raise which reduces the size of the maneuver needed to restore the apogee back to its initial value.

These factors arc pretty much i]] bal ante. For example, at about noon on 28 May ***2003*** the Moon will be at apogee at a radius of 406168 km. If a spacecraft starts in a circular parking orbit with a radius of 6500 km, then a 11 ohmann transfer to the Moon results in a veloci ty at the surface of the Moon of ***2504.4 m/s.*** Instead we can flyby to raise the perigee to a radius of 12000 km, since a 1 2000 km by 406168 km orbit has a period just 2/5 of the Moon's. 'Then a 6 m/s maneuver at perigee (where the velocity is 8033 m/s) lowers the velocity at the surface of the Moon to 2485.6 m/s for a net savings of about 13 m/s. Similarly, a 1/2 period intermediate orbit has a perigee at 78832 km and a perigee velocity of 2910 m/s. This case results in a net savings of 12 m/s after an 83 m/s perigee maneuver reduces the arrival surface velocity by 95 m/s to 2409.5 m/s,

The Moon, however, is so big relative to the Earth that this conic analysis is suspect. To verify these resul ts, two trajectories were integrated which included the effect of the central gravities of the Sun, Earth, and Moon. The first was a near-Hohmann transfer from a 6500 km radius parking orbit at Earth to an insertion into a 100 km altitude circular orbit at the Moon.

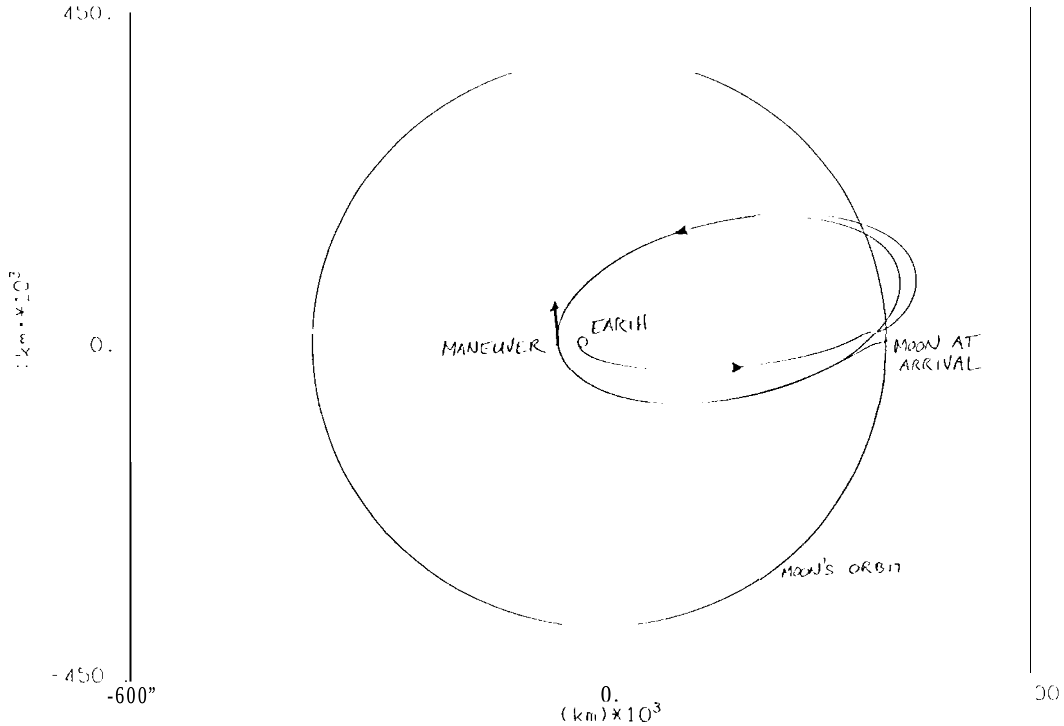

Figure 4 **Inertial lunar-orbit plane view of a reverse interior ΔV-Lunar-Gravity-Assist trajectory to the Moon.**

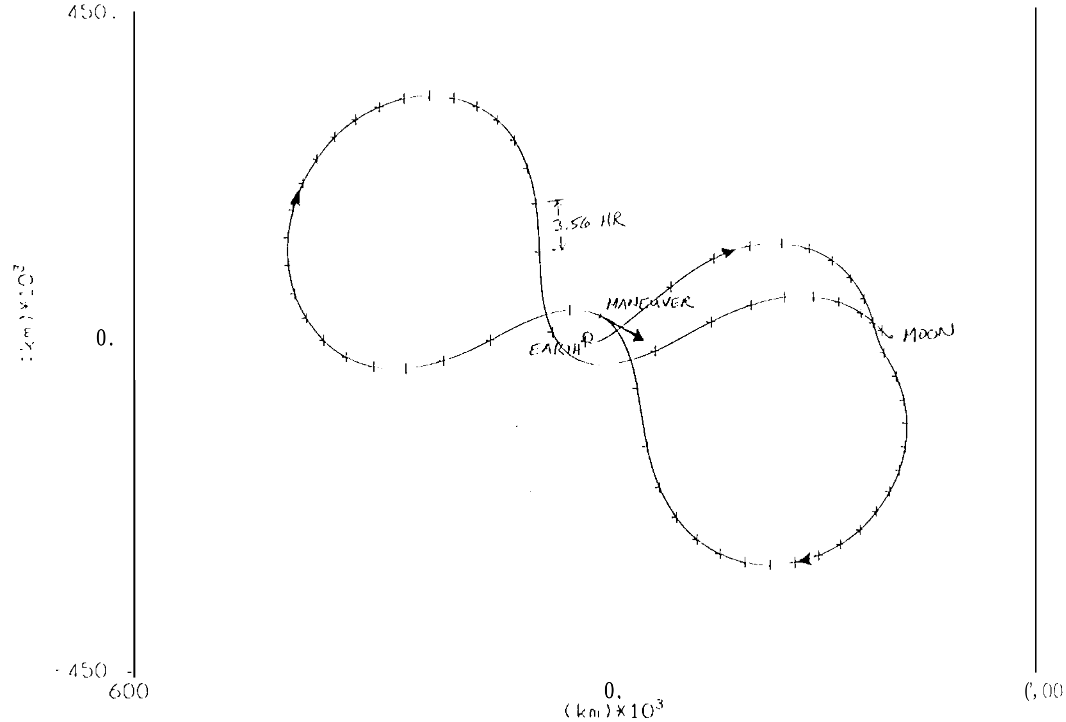

Figure 5 **Rotating lunar-orbit plane view (fixed Earth-Moon line) of the reverse interior ΔV-Lunar-Gravity -Assist trajectory shown in Figure 4.**

For this case, the injection ΔV ***was 3154.8*** m/s on ***93/8/29 11:04:03*** and the orbit insertion ΔV was 800.5 m/s on ***93/9/3*** 10:27:58, for a total ΔV of 3855.3 ***m/s.*** The second trajectory is illustrated in Figures 4 and 5. Figure 4 shows it in an inertial frame with the Moon's orbit also drawn. Figure 5 shows it in a rotating frame with the Earth-Moon line fixed. Since the flyby occurred on 93/9/3 13:47:27 at a distance of 17815 km from the Moon's center on the side towards the Earth, the injection ΔV was reduced slightly to 3150.4 m/s on 93/8/29 19:33:58. The post-flyby perigee ΔV was 13.3 m/s at a perigee radius of 32790 km on 93/9/13 13:08:04. The insertion ΔV was 768.6 m/s on 93/9/30 14:37:53, for a total ΔV of 3923.3 m/s and a net savings over the direct transfer of 23 m/s, about twice what the conic analysis predicted. Note also that the perigee distance is quite different than in the conic analysis, mostly caused by needing to allow for additional time spent in the orbit between the lunar flyby and the first perigee since the flyby delays the apogee as well as raising it.

This analysis demonstrates the existence of a new kind of lunar transfer trajectory. Although the ΔV savings over the Hohmann transfer are small, this transfer shows other advantages as well. Because the lunar flyby is distant and the post-flyby period is adjusted at perigee, course correction requirements are less demanding than for the direct transfer. Also, almost the entire trajectory is spent within the Moon's orbit so there are no longer-distance communication requirements.

SUGGESTIONS FOR FURTHER WORK

There are a number of open questions about ΔV-EGA trajectories suggested by this work. It is possible to calculate the ΔV gain without Jacobi's integral by finding the flight path angle change at Earth's orbit radius. How does this formula compare to the one in this paper? Can we learn anything new about Jacobi's integral from the comparison? What is the minimum exterior ΔV-EGA orbit leaving tangent to Earth's orbit which results in an energy gain? Is there a nearly one-year return ΔV-EGA trajectory (with a different eccentricity or inclination) which results in a ΔV gain? Is there a minimum departure C_3 necessary for a ΔV-EGA to result in a ΔV gain?

Several issues are open concerning reverse interior ΔV-LGA lunar transfers as well. First of all, the example shown in Figures 4 and 5 has not been optimized (note that the rendezvous ellipse is not tangent to the Moon's orbit). Also, when a multiconic model of the 2/5-period transfer was made it was found that the post-flyby orbit's apogee was inside the Moon's orbit, apparently removing the need for a perigee maneuver and thus preventing any AV savings. What happens *when a* transfer is attempted when the arrival does not happen at the M inn's apogee? H ow demanding are the real maneuver requirements? What if we include a model of the launch, so that the parking orbit at Earth has various inclinations with respect to the Moon's orbit so that the flyby has to reduce inclination as well as pump the orbit up? All these questions need to be answered in order to determine the practical usefulness of such a trajectory.

ACKNOWLEDGMENTS

The research described in this abstract was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration. I would like to thank Ralph Roncoli at the Jet Propulsion Laboratory for his careful reading and comments on earlier drafts of this paper.

REFERENCES

1. G. R. Hollenbeck, "New Flight Techniques for outer Planet Missions", AAS Paper 75-087, presented at the 1975 AAS/AIAA Astrodynamics Specialist Conference, Nassau, Bahamas, 28-30 July 1975.

2. R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, New York, 1987, pp. 372-373.

3. T. H. Sweetser, "An Estimate of the Global Minimum AV Needed for Earth-Moon Transfer", in J. K. Soldner, A, K. Misra, L. L. Sackett, and R. Holdaway (eds.), *Spaceflight Mechanics 1991, Volume 75,* Part 1 of *Advances in the Astronautical Sciences, pp.* 111-120, Univelt, San Diego, 1991.

4. T. H. Sweetser and K. Kiedron, "The Comet Rendezvous Asteroid Flyby Mission to Comet Kopff: Getting There is Half the Fun", in C. L. Thornton, R J. Proulx, J. E. Prussing, and F, R. Hoots (eds.), *Astrodynamics 1989*, Volume 71, Part II of *Advances in the Astronautical Sciences pp. 1133-1146,* Univelt, San Diego, 1990.

5. C.-W. L. Yen, "Ballistic Mercury orbiter Mission via Venus and Mercury Gravity Assists", AAS Paper 85-346, presented at the AAS/AIAA Astrodynamics Specialist Conference, Vail, Colorado, 12-35 August 1985.